

On the Complexity of some Geometrical Objects

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Abstract We recall the definition of the ϵ -distortion complexity of a set defined in [5] and the results obtained in this paper for Cantor sets of the interval defined by iterated function systems. We state an analogous definition for measures which may be more useful when dealing with dynamical systems. We prove a new lower bound in the case of Cantor sets of the interval defined by analytic iterated function systems. We also give an upper bound the ϵ -distortion complexity of invariant sets of uniformly hyperbolic dynamical systems.

1 Introduction.

It is common sense that some objects are simple like lines, circles, planes etc. In Mathematics and Physical sciences one also finds objects of different nature like fractal sets which are undoubtedly of more complicated nature. One can ask if it is possible to describe quantitatively this difference of complexity, namely can one define a number measuring the complexity of say a geometrical object. There are certainly many possible definitions. In section 2 we describe such a possible definition (see [5]).

A natural question is then to ask if this quantity is related to other Mathematical quantities measuring different properties of the object (dimension for example).

We now explain briefly some of the ideas behind our definition. In his seminal paper on information [14], Kolmogorov gave three possible definition for the quantity of information contained in a sequence of zeros and ones. They can also be viewed as a quantitative approach to the measure of complexity

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of such sequences. We will refer below to two definitions of Kolmogorov which can be briefly summarized as follows.

In one definition (algorithmic complexity, see for example [17]), a (finite) sequence is said to be complex if there does not exist a short computer program that generates this object. Quantitatively, the complexity is measured by the length of the smallest computer program that outputs this sequence. One can also use the total number of instructions executed by the machine to produce the sequence which may be much larger than the length of the program (due to loops for example).

In another definition of Kolmogorov (the combinatorial complexity, see for example [20]), the idea is that an object is complex if it is contained in a large set. Picking a particular object in a large set of equivalent objects require a large information. The opposite situation is even more obvious, if a set contains only one object, it is easy to pick up this object.

The definitions of Kolmogorov are for finite sequences of zeros and ones but we want to analyze continuous objects. One needs therefore some kind of discretization (see [2] for similar questions). One can think for example of drawing the object on a computer screen. In this case, very fine structures of the object are somehow irrelevant if their size is smaller than a pixel size. This leads naturally to the ideas that we have a fixed given precision $\epsilon > 0$, and the object will be described by a finite number of points covering the object up to a precision ϵ . More precisely, we will impose that the Hausdorff distance (see [18]) between the object and the finite set of points used for its description is smaller than ϵ .

Summarizing, we will start with a given reference frame with units, and look for finite set of points with rational coordinates whose Hausdorff distance to the object is smaller than ϵ . Finally we consider the programs generating the coordinates of these points. We finally optimize on the points positions and the program length. The optimal program length is what we call the distortion complexity of the object at precision ϵ .

We will also propose a similar definition for the complexity of measures which may be more adapted to the study of dynamical systems.

We refer to the book [3] which develops similar ideas with a different goal. See also [8], [6], [7], [4], [1] and references therein for related works.

2 Definitions and main results.

We first recall some definitions related to the Kolmogorov complexity (see [14] and [17]). We will denote by \mathcal{P} the set of finite sequences of zeros and ones. We will denote by $l(P)$ the length of the sequence P . An element $P \in \mathcal{P}$ will be considered as a program working on a computer (universal Turing machine).

We consider in \mathbb{R}^n a fixed orthogonal basis and a unit of length. We will consider below the programs whose output is a finite set of points in \mathbb{R}^n described by their coordinates (these are programs which terminate). For a program $P \in \mathcal{P}$ we will denote by $\text{out}(P)$ this finite set of points.

We also recall that the Hausdorff distance between two closed sets F and F' is defined by

$$d(F, F') = \max \left\{ \sup_{x \in F} d(x, F'), \sup_{y \in F'} d(y, F) \right\} .$$

This is a distance between closed sets (see [18] for more properties) which measures how the two sets differ.

We can now formulate our main definition, the idea is that given a precision ϵ , we look for the approximation of a set F by finitely many points which requires the smallest computer program.

Definition 1. For $\epsilon > 0$, the ϵ -distortion complexity of a closed set F (denoted by $\Delta_\epsilon(F)$) is the number

$$\Delta_\epsilon(F) = \min \{ l(P) \mid P \in \mathcal{P}, d(F, \text{out}(P)) \leq \epsilon \}$$

Our goal in the sequel is to understand how $\Delta_\epsilon(F)$ depends on ϵ for ϵ small and how it can be related to other quantitative properties of F , at least for some particular classes of sets. When there is no ambiguity on the set F , we will use the notation Δ_ϵ instead of $\Delta_\epsilon(F)$.

There is an easy upper bound for ϵ distortion complexity of a set F in terms of its box counting dimension d_F (see [9]). Given $\epsilon > 0$, we can cover the set by at most ϵ^{-d_F} balls of radius ϵ . We can use the centers of these balls to describe F at precision ϵ . To describe a point in a finite dimensional space we can give the dyadic expansion of its coordinates. This immediately leads to a bound

$$\Delta_\epsilon(F) \leq \mathcal{O}(1) \epsilon^{-d_F} \log \epsilon^{-1} . \quad (1)$$

We will see later that imposing some properties on the set F may substantially lower the ϵ -distortion complexity.

In the case of dynamical systems, instead of looking at the ϵ distortion complexity of an attractor, it may be interesting to look at the distortion complexity of a (invariant) probability measure. Recall that for probability measures on a compact set, the Kantorovich distance is a metric for the weak topology. We recall that (see [19]) it is defined by

$$d_K(\mu, \nu) = \inf_{f \in \mathcal{L}_1} \left(\int f d\mu - \int f d\nu \right) ,$$

where \mathcal{L}_1 is the set of Lipschitz continuous functions with Lipschitz constant at most one.

We can now consider programs whose output are atomic measures with a finite number of atoms and rational coefficients. In analogy with definition 1 we can define the ϵ -distortion complexity of a measure.

Definition 2. For $\epsilon > 0$, the ϵ -distortion complexity of a measure μ (denoted by $\Delta_\epsilon(\mu)$) is the number

$$\Delta_\epsilon(\mu) = \min \{l(\mathbf{P}) \mid \mathbf{P} \in \mathcal{P}, d_K(\mu, \text{out}(\mathbf{P})) \leq \epsilon\}$$

It is easy to verify that the ϵ -distortion complexity of the Lebesgue measure is bounded above by $\mathcal{O}(1) \log \epsilon^{-1}$. Since a better precision does not hurt, one can get a better bound by using a precision $1/n < \epsilon$ if the number n has a lower complexity.

Before we state our results, we recall the definition of Cantor sets in the interval $[0, 1]$ associated to iterated function systems.

Let \mathcal{I} be a finite set of indices with at least two elements. An (hyperbolic) *Iterated Function System* is a collection

$$\{\phi_i : A \rightarrow A : i \in \mathcal{I}\}$$

of contractions on A with uniform contraction rate (Lipschitz constant) $\rho \in (0, 1)$, and such that $\phi_i(A) \cap \phi_j(A) = \emptyset$ for $i \neq j$. We shall only consider hyperbolic iterated function systems with injective contractions (IHIFS for short).

If $\omega_0^\infty = (\omega_0, \omega_1, \dots)$ is an infinite sequence of indices (an element of $\mathcal{I}^\mathbb{N}$), the set

$$\mathcal{C} := \bigcup_{\omega_0^\infty \in \mathcal{I}^\mathbb{N}} \bigcap_{n=0}^{\infty} \phi_{\omega_n} \cdots \phi_{\omega_0}([0, 1])$$

is a Cantor set and satisfies

$$\mathcal{C} = \bigcup_{i \in \mathcal{I}} \phi_i(\mathcal{C}). \quad (2)$$

It contains all the accumulation points of the images of all the finite composition products of ϕ_i 's. We will use sometimes the notation $\mathcal{C}(\phi)$ to emphasize the collection of maps used to construct the Cantor set.

For $q \geq p$, we will denote by $\phi_{\omega_p^q}$ the map

$$\phi_{\omega_p^q} = \phi_{\omega_q} \circ \cdots \circ \phi_{\omega_p}.$$

If $p > q$ we define $\phi_{\omega_p^q}$ to be the identity.

The original Cantor set is obtained by using $\mathcal{I} = \{0, 1\}$, $\phi_0(x) = x/3$ and $\phi_1(x) = (2+x)/3$. It is easy to verify that the ϵ -distortion complexity of this Cantor set grows at most like $\log \epsilon^{-1}$. It was observed by A.Mandel that the ϵ -distortion complexity of this Cantor set grows as a function of ϵ^{-1} more slowly than any computable function.

We now state some of the results obtained in [5] in the case of Cantor sets.

Theorem 1. *Let \mathcal{C} be a Cantor set generated by an IHIFS with polynomial functions. Then*

$$\Delta_\epsilon(\mathcal{C}) \leq \mathcal{O}(\log(\epsilon^{-1})) .$$

Moreover, there exist (many) polynomial IHIFS's (with $\text{Card } \mathcal{J} = 2$) such that the generated Cantor set satisfies

$$\Delta_\epsilon(\mathcal{C}) \geq \delta(\log(\epsilon^{-1})) ,$$

for some $\delta > 0$

Theorem 2. *Let \mathcal{C} be a Cantor set generated by an IHIFS with real analytic functions. Then*

$$\Delta_\epsilon(\mathcal{C}) \leq \mathcal{O}((\log(\epsilon^{-1}))^2)$$

We will establish a converse result in section 3.

Theorem 3. *Let $k > 1$. For any $\delta > 0$, for any C^k Cantor set \mathcal{C} generated by an IHIFS with box counting dimension D , we have*

$$\Delta_\epsilon(\mathcal{C}) \leq \mathcal{O}(\epsilon^{-\frac{D}{k}-\delta})$$

Moreover, for any $\delta > 0$, there exist (many) C^k Cantor sets \mathcal{C} generated by an IHIFS (with $\text{Card } \mathcal{J} = 2$), with box counting dimension at most $D + \delta$, such that for any $\epsilon > 0$ small enough,

$$\epsilon^{-\frac{D}{k}+\delta} \leq \Delta_\epsilon(\mathcal{C}).$$

The word “many” in the second parts of Theorems 1 and 3 is given a precise (probabilistic) meaning in [5]. We will show in section 3 an analogous result in the analytic case where this word means large cardinality (the combinatorial complexity in the sense of Kolmogorov's paper [14]).

Here we want to emphasize the difference of behavior between Cantor sets defined by analytic IHIFS and Cantor sets defined by differentiable IHIFS as appearing in Theorems 2 and 3 respectively. In the first case the upper bound grows rather slowly while in the second case, the growth is much faster and depends crucially on two properties of the system, the box counting dimension of the set and the regularity of the IHIFS.

These results (and some others) are proved in details in [5]. In the next sections we will explain some of the ideas behind these proofs applied to two new results. In section 3 we will prove a lower bound in the analytic case, and in section 4 we will prove an upper bound for the ϵ -distortion complexity of the attractor of a uniformly hyperbolic dynamical system.

3 The ϵ -distortion complexity of real analytic IHIFS.

In this section we will obtain some results for Cantor sets of the interval defined by analytic IHIFS. The results are formulated in terms of the so called combinatorial complexity (see [14] and [15]), namely we will “count” the number of Cantor sets with a given property. The relation with Kolmogorov complexity is that since the number of programs of length p is at most 2^p , if you have a set with larger cardinality, some of its elements should have complexity larger than p .

We start by introducing some definitions and notations. We choose once for all a number $R > 1/2$ and for any $0 < \tilde{\rho} < \rho < 1/2$, $Q > 1 + R/2$, we denote by $\mathcal{H}_R(\tilde{\rho}, \rho, Q)$ the set of functions f satisfying

- i) f is analytic and has modulus bounded by Q in the complex disk D_R centered at $z = 1/2$ and with radius R .
- ii) f maps the interval $[0, 1]$ into itself, and satisfies

$$\tilde{\rho} \leq \inf_{x \in [0,1]} f'(x) \leq \sup_{x \in [0,1]} f'(x) \leq \rho .$$

Note that this set is non empty since it contains the function $f(x) = (\rho + \tilde{\rho})(x - 1/2)/2 + 1/2$. We will use the indices 0 and 1 to specify the components of the elements of $\mathcal{H}_R(\tilde{\rho}, \rho, Q)^2$. We will denote by $\mathcal{K}_R(\tilde{\rho}, \rho, Q)$ the subset of $\mathcal{H}_R(\tilde{\rho}, \rho, Q)^2$ given by

$$\mathcal{K}_R(\tilde{\rho}, \rho, Q) = \{ \phi = (\phi_0, \phi_1) \in \mathcal{H}_R(\tilde{\rho}, \rho, Q)^2, \phi_0(0) = 0, \phi_1(1) = 1 \} .$$

Given two Cantor sets \mathcal{C} and \mathcal{C}' , we will say that they are ϵ -separated if

$$d(\mathcal{C}, \mathcal{C}') > \epsilon .$$

For $\epsilon > 0$, we will denote by $\mathcal{N}(\epsilon, \tilde{\rho}, \rho, Q, R)$ the maximal number of pairwise ϵ -separated Cantor sets defined by analytic IHIFS with $\text{Card } \mathcal{S} = 2$ and $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$. This is the $\epsilon/2$ capacity as defined in [15].

Before we give the main result of this section which is new with respect to [5], and complements Theorem 2, we prove some technical lemmas.

Lemma 1. *For any $\phi^0 \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, if g_0 and g_1 are analytic and bounded in D_R and satisfy $g_0(0) = g_1(1) = 0$, and*

$$\sup_{x \in [0,1]} |g'_0(x)| + \sup_{x \in [0,1]} |g'_1(x)| < \inf \{ \tilde{\rho}, 1/2 - \rho \}$$

then $\phi = (\phi_0^0 + g_0, \phi_1^0 + g_1)$ satisfies

$$\sup_{x \in [0,1]} \sup_{\omega_0^n \in \{0,1\}^{n+1}} |\phi_{\omega_0^n}(x) - \phi_{\omega_0^n}^0(x)| \leq \frac{1}{1 - \rho} \max \left\{ \sup_{x \in [0,1]} |g_0(x)|, \sup_{x \in [0,1]} |g_1(x)| \right\}$$

Proof. The proof is recursive. \square

For any integer n , we will denote by $\mathcal{C}_n(\phi)$ the set

$$\mathcal{C}_n(\phi) = \bigcup_{\omega_0^{n-1} \in \{0,1\}^n} \phi_{\omega_0^{n-1}}(0) \bigcup_{\omega_0^{n-1} \in \{0,1\}^n} \phi_{\omega_0^{n-1}}(1)$$

Note that this set has cardinality $2^n + 2$.

Lemma 2. *For any integer $n \geq 2$ we have*

$$\mathcal{C}_n(\phi) \setminus \mathcal{C}_{n-1}(\phi) = \bigcup_{\omega_0^{n-2} \in \{0,1\}^{n-1}} \phi_{\omega_0^{n-2}1}(0) \bigcup_{\omega_0^{n-2} \in \{0,1\}^{n-1}} \phi_{\omega_0^{n-2}0}(1)$$

Proof. The proof is recursive. \square

Lemma 3. *Let $(\xi_u)_{u \in \mathcal{C}_n(\phi)}$ be a (finite) sequence of complex numbers. Then for any $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, the polynomial*

$$h(z) = \sum_{u \in \mathcal{C}_n(\phi)} \xi_u \prod_{\substack{y \in \mathcal{C}_n(\phi) \\ y \neq u}} \frac{z - y}{u - y}$$

satisfies

$$\sup_{z \in D_R} |h(z)| + \sup_{z \in D_R} |h'(z)| \leq 16 \cdot 4^n \left(\frac{1}{2} + R \right)^{2^n+2} \tilde{\rho}^{-8 \cdot 2^n} \sup_{z \in \mathcal{C}_n(\phi)} |\xi_z|.$$

Proof. If two points u belongs to $\mathcal{C}_n(\phi)$, the nearest point in $\mathcal{C}_n(\phi)$ is at distance at least $\tilde{\rho}^n$, the two next nearest neighbor are at distance at least $\tilde{\rho}^{n-1}$ etc. The result follows. \square

Lemma 4. *Let $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$. For any integer n , when ω_0^{n-1} varies in $\{0,1\}^n$, the intervals*

$$[\phi_{\omega_0^{n-1}}(0), \phi_{\omega_0^{n-1}}(1)]$$

are pairwise disjoint, and their pairwise distance is at least $\tilde{\rho}^n(1 - 2\rho)$. Their union contains the Cantor set $\mathcal{C}(\phi)$. Moreover, the points in $\mathcal{C}_n(\phi)$ belong to $\mathcal{C}(\phi)$.

Proof. The proof follows easily recursively from the fact that $\rho < 1/2$, and the point 0 (respectively 1) is fixed by the map ϕ_0 (respectively ϕ_1). \square

The following Lemma is a particular version of Lemma 4.1 in [5] ($I = I' = [0, 1]$ in the notations of this paper), and will be used to get a lower bound on the Hausdorff distance of two closed sets F and F' with holes H and H' .

Lemma 5. *Let F and F' be two closed subsets of $[0, 1]$. Let $H = [c, d]$ and $H' = [c', d']$ be closed sub-intervals of $]0, 1[$. Assume that*

$$\{c, d\} \subset F, \{c', d'\} \subset F', F \cap \overset{\circ}{H} = \emptyset, F' \cap \overset{\circ}{H'} = \emptyset.$$

Moreover, assume that for some $\epsilon > 0$

$$|c - d| > 2\epsilon, |c' - d'| > 2\epsilon, \max\{|c - c'|, |d - d'|\} > \epsilon.$$

Then

$$d(F, F') > \epsilon.$$

The proof follows at once from the definition of the Hausdorff distance.

We see from this result and Lemma 4 that in order to construct “many” Cantor sets it is enough to construct the points $\phi_{\omega_0^{k-1}}(0)$ and $\phi_{\omega_0^{k-1}}(1)$. However since we want these points to be generated by analytic maps there are some constraints.

The following result provides an upper bound for $\mathcal{N}(\epsilon)$

Theorem 4. *For any R, ρ and $\tilde{\rho}$ and Q as above, there exists a constant $K = K(R, \tilde{\rho}, \rho, Q) > 0$ such that for any $1/2 > \epsilon > 0$*

$$\mathcal{N}(\epsilon) \leq K (\log \epsilon)^2.$$

Proof. The proof is analogous to the proof of Theorem 2 (see [5]) but we sketch it for the convenience of the reader. Let $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, we have for any z in the interior of D_R

$$\phi_0(z) = \sum_{n=0}^{\infty} \phi_0^{(n)} \left(z - \frac{1}{2} \right)^n$$

with

$$|\phi_0^{(n)}| \leq \frac{Q}{R^n}.$$

A similar expression and estimate hold for ϕ_1 . The number $\phi_0^{(n)}$ is $n!$ times the n^{th} derivative of ϕ_0 in $z = 1/2$. Let

$$N = \left\lceil \frac{\log(16Q/[(2R-1)\epsilon(1-\rho)])}{\log(2R)} \right\rceil$$

and define

$$\tilde{\phi}_0(z) = \sum_{n=0}^N \phi_0^{(n)} \left(z - \frac{1}{2} \right)^n, \quad \tilde{\phi}_1(z) = \sum_{n=0}^N \phi_1^{(n)} \left(z - \frac{1}{2} \right)^n.$$

We have

$$\sup_{x \in [0,1]} |\phi_0(x) - \tilde{\phi}_0(x)| \leq \sum_{n=N+1}^{\infty} \frac{Q}{(2R)^n} = \frac{Q}{(2R)^N(2R-1)} \leq \frac{\epsilon(1-\rho)}{16}.$$

We define for each $0 \leq n \leq N$ and $\epsilon > 0$ the finite set

$$\mathcal{B}_{\epsilon,n} = \left\{ (p + iq) \in (1 - \rho)/16, -\frac{17Q}{(2R)^n \epsilon (1 - \rho)} \leq p, q \leq \frac{17Q}{(2R)^n \epsilon (1 - \rho)} \right\}.$$

Let $\mathcal{Q}_{\epsilon,N}$ denote the set of finite sequences of complex numbers given by

$$\mathcal{Q}_{\epsilon,N} = \{(\xi_r)_{0 \leq r \leq N}, \xi_r \in \mathcal{B}_{\epsilon,r}, \forall 0 \leq r \leq N\}$$

To each finite sequence of complex numbers $\underline{\xi}$ in $\mathcal{Q}_{\epsilon,N}$, we associate the polynomial

$$f_{\underline{\xi}}(z) = \sum_{r=0}^N \xi_r \left(z - \frac{1}{2}\right)^r.$$

It is easy to verify that for any $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, there is a sequence $\underline{\xi} \in \mathcal{Q}_{\epsilon,N}$ such that

$$\sup_{x \in [0,1]} |f_{\underline{\xi}}(x) - \phi_0(x)| < \frac{(1 - \rho)\epsilon}{8}.$$

A similar estimate holds for ϕ_1 (with in general another sequence $\underline{\xi}$). This implies that we can find a collection \mathcal{G}_ϵ of elements of $\mathcal{K}_R(\tilde{\rho}, \rho, Q)$ with cardinality at most

$$\text{Card}(\mathcal{G}_\epsilon) \leq \text{Card}(\mathcal{Q}_{\epsilon,N})^2 \leq \left(\frac{34Q}{\epsilon(1 - \rho)}\right)^N,$$

such that for any $\phi \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, we can find a $\tilde{\phi} \in \mathcal{G}_\epsilon$ such that

$$\max \left\{ \sup_{x \in [0,1]} |\phi_0(x) - \tilde{\phi}_0(x)|, \sup_{x \in [0,1]} |\phi_1(x) - \tilde{\phi}_1(x)| \right\} \leq \frac{(1 - \rho)\epsilon}{4}.$$

This also follows directly from results in [15].

From Lemmas 1 and 4, we have

$$d(\mathcal{C}(\phi), \mathcal{C}(\tilde{\phi})) \leq \epsilon.$$

This finishes the proof of the Theorem. \square

Proposition 1. *Given $R, \tilde{\rho}, \rho$ and Q as above, there exists $\tilde{\epsilon} = \epsilon(R, \tilde{\rho}, \rho, Q) > 0$ such that if $\phi^0 \in \mathcal{K}_R(\tilde{\rho}, \rho, Q)$, $0 < \epsilon_0 < \tilde{\epsilon}$, and*

$$0 < \epsilon_1 < \epsilon_0^{5+2 \log_2(R+1/2)-16 \log_2 \tilde{\rho}},$$

there exists a subset $\mathcal{Q} = \mathcal{Q}(R, \tilde{\rho}, \rho, Q, \phi^0, \epsilon_0, \epsilon_1)$ of $\mathcal{K}_R(\tilde{\rho} - \epsilon_0, \rho + \epsilon_0, Q + \epsilon_0)$ with cardinality at least $e^{2^{-4} \log(\epsilon_0/\epsilon_1) \log \epsilon_0^{-1}}$ such that for any $\phi \in \mathcal{Q}$

$$d(\mathcal{C}(\phi^0), \mathcal{C}(\phi)) \leq \frac{\epsilon_0}{3},$$

and

$$\sup_{z \in D_R} |\phi_0^0(z) - \phi_0(z)| + \sup_{z \in D_R} |\phi_1^0(z) - \phi_1(z)| \leq \frac{\epsilon_0}{4}.$$

Moreover, for any $\phi \neq \phi'$ in \mathcal{Q}

$$d(\mathcal{C}(\phi), \mathcal{C}(\phi')) \geq \epsilon_1.$$

Proof. For $\epsilon_0 > 0$ we define the integer $N = N(\epsilon_0^{-1})$ by

$$N = \lfloor \log_2 \log(\epsilon_0^{-1}) \rfloor.$$

In other words

$$\frac{1}{2} \log(\epsilon_0^{-1}) \leq 2^N \leq \log(\epsilon_0^{-1}).$$

We will denote by \mathcal{J}_N the set of maps from $\{0, 1\}^{N-2}$ to $\{1, 2, \dots, \lfloor \sqrt{\epsilon_0/\epsilon_1} \rfloor\}^4$.

For an element $J \in \mathcal{J}_N$, we define a pair of functions (g_0^J, g_1^J) as follows. We start by defining the functions on the set $\mathcal{C}_{N-1}(\phi^0)$ by setting

$$g_0^J(x) = g_1^J(x) = 0$$

for any $x \in \mathcal{C}_{N-1}(\phi^0)$.

We next define the the two functions on $\mathcal{C}_N(\phi^0) \setminus \mathcal{C}_{N-1}(\phi^0)$. Using Lemma 2, this is done by setting for any $\omega_0^{N-2} \in \{0, 1\}^{N-1}$

$$g_0^J(\phi_{\omega_0^{N-2}1}^0(0)) = -J_1(\omega_0^{N-2}) \epsilon_1, \quad g_0^J(\phi_{\omega_0^{N-2}0}^0(1)) = J_2(\omega_0^{N-2}) \epsilon_1,$$

$$g_1^J(\phi_{\omega_0^{N-2}1}^0(0)) = -J_3(\omega_0^{N-2}) \epsilon_1, \quad g_1^J(\phi_{\omega_0^{N-2}0}^0(1)) = J_4(\omega_0^{N-2}) \epsilon_1.$$

Finally, we define the functions as polynomials by the Lagrange interpolation formula

$$g_0^J(x) = \sum_{z \in \mathcal{C}_N(\phi^0)} g_0^J(z) \prod_{\substack{y \in \mathcal{C}_N(\phi^0) \\ y \neq z}} \frac{x - y}{z - y},$$

and similarly for g_1^J .

From the definition of N and the condition on ϵ_1 we get (for $\tilde{\epsilon}$ small enough) from Lemma 3 and for any $J \in \mathcal{J}_N$

$$\begin{aligned} & \sup_{z \in D_R} |g_0^J(z)| + \sup_{z \in D_R} |g_1^J(z)| + \sup_{z \in D_R} |g_0^J(z)| + \sup_{z \in D_R} |g_1^J(z)| \\ & \leq 42^N \left(\frac{1}{2} + R \right)^{2^N} \tilde{\rho}^{-8 \cdot 2^N} \rho^N \sqrt{\epsilon_0 \epsilon_1} \leq \frac{\epsilon_0}{4}, \end{aligned}$$

which implies $(\phi_0^0 + g_0^J, \phi_1^0 + g_1^J) \in \mathcal{K}_R(\tilde{\rho} - \epsilon_0, \rho + \epsilon_0, Q + \epsilon_0)$.

Using Lemmas 4, 5, and $\tilde{\rho}^N > \epsilon_1(1 - 2\rho)$, we conclude that if J and J' are two different elements of \mathcal{J}_N , we have

$$d(\mathcal{C}(\phi_0^0 + g_0^J, \phi_1^0 + g_1^J), \mathcal{C}(\phi_0^0 + g_0^{J'}, \phi_1^0 + g_1^{J'})) > \epsilon_1 .$$

Using Lemmas 4 and 1, we conclude that (for ϵ_0 small enough), for any $J \in \mathcal{J}_N$

$$d(\mathcal{C}(\phi^0), \mathcal{C}(\phi_0^0 + g_0^J, \phi_1^0 + g_1^J)) \leq \frac{\sqrt{\epsilon_0 \epsilon_1}}{1 - \rho} < \frac{\epsilon_0}{3} .$$

The proposition follows from the estimate

$$\text{Card}(\mathcal{J}_N) = \left\lfloor \sqrt{\frac{\epsilon_0}{\epsilon_1}} \right\rfloor^{2^{N-2}} .$$

□

We can now state and prove the main result of this section which complements Theorem 2.

Theorem 5. *There exists Cantor sets \mathcal{C} defined by analytic IHIFS such that*

$$\liminf_{\epsilon \rightarrow 0} \frac{\Delta_\epsilon(\mathcal{C})}{(\log \epsilon^{-1})^2} > 0 .$$

Proof. Let $R, \tilde{\rho}_0, \rho_0$ and Q_0 satisfy the assumptions defined at the beginning of this section. We will use the sequence of numbers

$$\epsilon_p = 2^{-K} \gamma^p ,$$

with $K > 0$ (large enough) and $\gamma > 1$ to be chosen later on. Let

$$\rho_{p+1} = \rho_p + \epsilon_p, \tilde{\rho}_{p+1} = \tilde{\rho}_p - \epsilon_p, Q_{p+1} = Q_p + \epsilon_p .$$

We assume that K is large enough so that

$$\frac{\tilde{\rho}_0}{2} < \inf_p \tilde{\rho}_p \leq \sup_p \rho_p < \frac{2\rho_0 + 1}{4} .$$

Let $V > 0$ and $\Gamma > 0$ to be chosen later on. We consider the recursive assumption:

For any integer $p \geq 0$, there exists a subset \mathcal{E}_p of $\mathcal{K}_R(\tilde{\rho}_p, \rho_p, Q_p)$, with the following properties.

1) The cardinality of \mathcal{E}_p is at least

$$2^{V(\log_2 \epsilon_p)^2} .$$

2) For any $0 \leq \ell \leq p$

$$\inf_{\phi \in \mathcal{E}_p} \Delta_{\epsilon_\ell}(\mathcal{C}(\phi)) \geq \Gamma(\log_2 \epsilon_\ell)^2 .$$

3) For any $\phi \neq \phi' \in \mathcal{E}_p$,

$$d(\mathcal{C}(\phi), \mathcal{C}(\phi')) > \epsilon_p .$$

4) If $p > 0$, for any $\phi \in \mathcal{E}_p$, there exists $\tilde{\phi} \in \mathcal{E}_{p-1}$ such that

$$d(\mathcal{C}(\tilde{\phi}), \mathcal{C}(\phi)) \leq \frac{\epsilon_{p-1}}{3} ,$$

and

$$\sup_{z \in D_R} |\tilde{\phi}_0(z) - \phi_0(z)| + \sup_{z \in D_R} |\tilde{\phi}_1(z) - \phi_1(z)| \leq \epsilon_{p-1} .$$

For $p = 0$, we can find $\Gamma > 0$ and $V > 0$ depending on K such that for any $K > 0$ large enough, the above assumption is satisfied.

We now prove that we can choose the constants K, γ, Γ, V , such that if the assumption is true at step $p \geq 0$, it will also be true at step $p + 1$.

We impose the lower bound

$$\gamma > 5 + 2 \log_2(R + 1/2) - 16 \log_2(\tilde{\rho}_0/2) .$$

This implies for any $p \geq 0$

$$\gamma > 5 + 2 \log_2(R + 1/2) - 16 \log_2(\tilde{\rho}_p) .$$

To each element $\phi^0 \in \mathcal{E}_p$, we apply Proposition 1 with ϵ_0 replaced by ϵ_p and ϵ_1 replaced by ϵ_{p+1} .

We obtain a finite subset \mathcal{X}_p of $\mathcal{K}_R(\tilde{\rho}_{p+1}, \rho_{p+1}, Q_{p+1})$ with cardinality

$$2^{2^{-4}(\log \epsilon_{p+1} - \log \epsilon_p) \log \epsilon_p + V(\log_2 \epsilon_p)^2} .$$

Assume $\phi \neq \phi' \in \mathcal{X}_p$ come from the application of Proposition 1 to the same $\phi^0 \in \mathcal{E}_p$, then

$$d(\mathcal{C}(\phi), \mathcal{C}(\phi')) > \epsilon_{p+1} .$$

If they come from $\phi^0 \in \mathcal{E}_p$ and $\tilde{\phi}^0 \in \mathcal{E}_p$ respectively with $\phi^0 \neq \tilde{\phi}^0$, we have from the triangle inequality and the recursive assumption (and K large enough)

$$\begin{aligned} & d(\mathcal{C}(\phi), \mathcal{C}(\phi')) \\ & \geq d(\mathcal{C}(\phi^0), \mathcal{C}(\tilde{\phi}^0)) - d(\mathcal{C}(\phi), \mathcal{C}(\phi^0)) - d(\mathcal{C}(\phi'), \mathcal{C}(\tilde{\phi}^0)) \\ & \geq \epsilon_p - 2 \frac{\epsilon_p}{3} \geq \epsilon_{p+1} . \end{aligned}$$

Therefore, for each $\phi \neq \phi' \in \mathcal{X}_p$ we have

$$d(\mathcal{C}(\phi), \mathcal{C}(\phi')) > \epsilon_{p+1} ,$$

which is the third part of the recurrence assumption.

Since the number of programs of length $\Gamma(\log_2 \epsilon)^2$ is at most $2^{\Gamma(\log_2 \epsilon)^2}$, there is a subset \mathcal{E}_{p+1} of \mathcal{X}_p with cardinality at least

$$\begin{aligned} & 2^{2^{-8}(\log \epsilon_{p+1} - \log \epsilon_p) \log \epsilon_p + V(\log_2 \epsilon_p)^2} - \sum_{\ell=0}^{p+1} 2^{\Gamma(\log_2 \epsilon_\ell)^2} \\ & \geq 2^{2^{-8}(\log \epsilon_{p+1} - \log \epsilon_p) \log \epsilon_p + V(\log_2 \epsilon_p)^2} - (p+2) 2^{\Gamma(\log_2 \epsilon_{p+1})^2} \end{aligned}$$

such that for all $0 \leq j \leq p+1$

$$\Delta_{\epsilon_j}(\mathcal{C}(\phi)) > \Gamma(\log_2 \epsilon_j)^2.$$

We now choose the numbers γ , Γ and V such that for any $p \geq 0$

$$2^{2^{-8}(\log \epsilon_{p+1} - \log \epsilon_p) \log \epsilon_p + V(\log_2 \epsilon_p)^2} > 2^{V(\log_2 \epsilon_{p+1})^2} + (p+2) 2^{\Gamma(\log_2 \epsilon_{p+1})^2}.$$

This will prove the first and second parts of the recursive assumption. Observing that $\epsilon_{p+1} = \epsilon_p^\gamma$, it is enough to have the above inequality to ensure that together with K large enough, we have

$$2^{-8}(\gamma - 1)(\log \epsilon_p)^2 > 2\gamma^2(V + \Gamma)(\log \epsilon_p)^2.$$

For any given $\gamma > 1$ this can be satisfied by taking for example

$$0 < V = \Gamma < \frac{2^{-8}(\gamma - 1)}{5\gamma^2}.$$

The last part of the recursive assumption follows directly from Proposition 1.

It is easy to verify that the sequence of sets $\mathcal{K}_R(\tilde{\rho}_p, \rho_p, Q_p)$ is increasing. Moreover, these are closed sets for the sup norm on analytic functions in D_R , and compact for the sup norm on analytic functions in $D_{R'}$ for any $R' < R$ (by Montel's Theorem).

Let \mathcal{E} be the set of accumulation points of sequences in (\mathcal{E}_p) . It is easy to verify from the last part of the recursion assumption, that for γ large enough, if $\phi \in \mathcal{E}$, for any $p \geq 0$, there exists $\phi^{(p)} \in \mathcal{E}_p$ such that

$$d(\mathcal{C}(\phi^{(p)}), \mathcal{C}(\phi)) \leq \frac{1}{3} \sum_{\ell=p}^{\infty} \epsilon_\ell \leq \frac{\epsilon_p}{2}.$$

Let $\phi \in \mathcal{E}$, and $0 < \epsilon < \epsilon_2$, there is a unique p such that

$$\frac{\epsilon_{p+1}}{2} < \epsilon \leq \frac{\epsilon_p}{2}.$$

Let $\tilde{\phi} \in \mathcal{E}_p$ be such that

$$d(\mathcal{C}(\tilde{\phi}), \mathcal{C}(\phi)) \leq \frac{\epsilon_p}{2}$$

It follows at once from the definition of the ϵ -distortion complexity that

$$\Delta_{\epsilon_p/2}(\mathcal{C}(\phi)) \geq \Delta_{\epsilon_p}(\mathcal{C}(\tilde{\phi})) \geq \Gamma(\log \epsilon_p)^2 \geq \Gamma \gamma^{-2} (\log \epsilon)^2.$$

Since this holds for any $\epsilon > 0$, the theorem is proved. \square

We observe that from the proof of the previous Theorem one can derive a lower bound on the ϵ -entropy (see [15], or in other words a lower bound on the combinatorial complexity (see [14]), namely how many Cantor sets generated by analytic IHIFS are there which differ at precision ϵ .

4 An upper bound for dynamical systems.

In this section we prove an upper bound in the case of uniformly hyperbolic diffeomorphisms of compact subsets of \mathbb{R}^d .

Theorem 6. *Let f be a C^k ($k \geq 1$) diffeomorphism of a compact subset M of \mathbb{R}^d ($d \geq 2$). Let \mathcal{A} denote an attractor of f . Assume that on a neighborhood \mathcal{V} of \mathcal{A} the diffeomorphism f is uniformly hyperbolic (see [11]) and denote by $\lambda_- < 0 < \lambda_+$ two numbers such that*

$$\sup_{x \in \mathcal{V}} \|D_x f\| \leq e^{\lambda_+},$$

and for any $y \in \mathcal{V}$,

$$d(f(y), \mathcal{A}) \leq e^{\lambda_-} d(y, \mathcal{A}).$$

Then for ϵ small enough, the ϵ -distortion complexity of \mathcal{A} is bounded by

$$\Delta_\epsilon(\mathcal{A}) \leq \mathcal{O}(1) \epsilon^{-d_{\mathcal{A}}(\lambda_+ - \lambda_-/k)/(\lambda_+ - \lambda_-)} (\log \epsilon^{-1})^2,$$

where $d_{\mathcal{A}}$ denotes the box counting dimension of \mathcal{A} .

Note that if we take $\lambda_- = 0$ we recover the trivial upper bound (1), and similarly for $k = 1$. The number $\lambda_- < 0$ can be easily related to the linear expansion and contraction factors of the unstable and stable linear bundles (see [11]).

Proof. We give a sketch of the proof. Let $\delta > 0$ to be chosen later on. For $\epsilon > 0$ small enough, we need of the order of $\epsilon^{-\delta d_{\mathcal{A}}}$ balls of radius ϵ^δ to cover the attractor \mathcal{A} . We make a choice of such a covering and denote by \mathcal{C} the sets of centers of the balls.

Let m be an integer to be chosen later on. For any point x we have

$$f^m(x + y) = f^m(x) + Df^m(x)(y) + D^2 f^m(x)(y, y)/2 + \dots$$

$$+ D^{k-1} f^m(x)(y, \dots, y)/(k-1)! + \mathcal{O}(\|y\|^k \|D^k f^m\|) .$$

By the chain rule $\|D^k f^m\|$ is at most of order $m^k \exp(m k \lambda_+)$. We impose the two conditions

$$e^{m k \lambda_+} \epsilon^{k\delta} < \epsilon , \quad \text{and} \quad e^{m \lambda_-} \epsilon^\delta < \epsilon .$$

Eliminating m between the two conditions, one gets

$$\delta \geq \frac{\lambda_+ - \lambda_-/k}{\lambda_+ - \lambda_-} ,$$

and we will use this minimal value of δ . Note that we can take $m = \mathcal{O}(\log \epsilon^{-1})$.

We need to describe for each $x \in \mathcal{C}$ the quantities $f^m(x)$, $Df^m(x)$ etc. up to $D^{k-1} f^m(x)$, at a precision of order ϵ . This gives a complexity bounded above by $\mathcal{O}(1) m \log(\exp(m k \lambda_+) \epsilon^{-1} m^k)$.

Finally, in order to describe the attractor, we can start with a regular lattice $\mathcal{L}_{m,\epsilon}$ with lattice size $\epsilon \exp(-m \lambda_+)$. Describing this lattice requires only a complexity of at most $-\mathcal{O}(1) \log(\epsilon \exp(-m \lambda_+))$ since this is a regular lattice.

For each $x \in \mathcal{C}$ we consider the set $f^m(\mathcal{L}_{m,\epsilon} \cap B_{\epsilon^\delta}(x))$. All the points in this set are within distance ϵ of \mathcal{A} since $\epsilon^\delta e^{m \lambda_-} \leq \epsilon$.

Moreover, for any $z \in \mathcal{A}$, $f^{-m}(z) \in \mathcal{A}$ and therefore there is an $x \in \mathcal{C}$ such that

$$d(f^{-m}(z), x) \leq \epsilon^\delta .$$

Therefore, there exists a point $\tilde{y} \in \mathcal{L}_{m,\epsilon} \cap B_{\epsilon^\delta}(x)$ such that

$$d(f^{-m}(z), \tilde{y}) \leq \epsilon \exp(-m \lambda_+)$$

implying

$$d(z, f^m(\tilde{y})) \leq \epsilon .$$

Therefore

$$d(\mathcal{A}, \cup_{x \in \mathcal{C}} (\mathcal{L}_{m,\epsilon} \cap B_{\epsilon^\delta}(x))) \leq \epsilon ,$$

and the result follows. \square

5 Remarks and open questions.

In this section we state some open problems which naturally arise from the previous results.

5.1 *Some questions about Cantor sets.*

In the proof of Theorem 5, in order to prove that there exists a large enough collection of Cantor sets satisfying the lower bound, we constructed many polynomials. These polynomials are of course analytic and even entire functions, but the whole collection cannot be considered from the point of view of Theorem 1 because their degree depends on ϵ (it is of order $\log \epsilon^{-1}$). This raises the question of understanding better this construction. One can try to use instead of the Lagrange interpolation formula the Carleson interpolation formula (see [10] for example). I expect this may improve the constant in front of the $(\log \epsilon^{-1})^2$ but a more interesting question would be to understand where data like the dimension appear in the asymptotic behavior of the ϵ -distortion complexity when ϵ tends to zero (in the prefactor?).

In another direction, one may ask how to fill the gap between the $\log \epsilon^{-1}$ in Theorem 1 and the $(\log \epsilon^{-1})^2$ in Theorems 2 and 5. A natural candidate would be to look at entire functions of various order (see for example [16] for definition and results). Between the $(\log \epsilon^{-1})^2$ behavior for analytic functions and the behavior as a power of ϵ^{-1} for C^k functions (Theorems 2, 5 and 3), one can try to fill the gap by looking at quasi-analytic functions (see for example [16] for definition and results).

The proof of Theorem 3 in [5] was based on the use of the scaling function for fractal sets. Is it possible to give a proof based on direct interpolation as we did for Theorem 5?

Some of the results in [5] can probably be extended to Cantor sets in higher dimension. The question of the complexity of measures is essentially untouched as far as I know. Is it related to other quantities like dimension and capacity? (see for example [12] for definitions and properties).

Note also that it is not clear if we can get generic (or prevalent) results (see [13] for definitions), these would have to be formulated in the Hausdorff metric of the set of Cantor sets. We do not expect to have such properties from the point of view of the set of IHIFS. More precisely, it seems possible for example that the set of real analytic IHIFS leading to a Cantor set with ϵ -distortion complexity bounded above by $\mathcal{O}(1)(\log \epsilon^{-1})^{2-\sigma}$ for some $\sigma > 0$ for any ϵ small enough is of second category.

5.2 *Some questions about dynamical systems.*

The estimate in Theorem 6 can be easily extended to Riemannian manifolds, but is of course very rough and one would like to use Lyapunov exponents instead of uniform bounds. The result should then involve invariant measures. We formulate a conjecture in this direction.

Conjecture. Let μ be an SRB measure for a C^k diffeomorphism f of a compact surface with Lyapunov exponents $\lambda_- < 0 < \lambda_+$, and dimension d_μ . Then

$$\limsup_{\epsilon \rightarrow 0} \frac{\log \Delta_{\epsilon}(\mu)}{\log \epsilon^{-1}} \leq d_{\mu} \frac{\log \lambda_+ - \frac{\log \lambda_-}{k}}{\log \lambda_+ - \log \lambda_-}.$$

We also conjecture that the above bound is saturated for many diffeomorphisms (in an adequate sense).

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